Analyse vectorielle

I. Opérateurs

définition intrinsèque des opérateurs	
gradient et différentielle	$df = \overrightarrow{grad}\, f \cdot \overrightarrow{d\ell}$
divergence et flux	$d\phi_{\vec{A}} = (\mathrm{div}\, \vec{A})\, d\tau$
rotationnel et circulation	$dC = (\overrightarrow{rot}\, \vec{A}) \cdot \overrightarrow{dS}$
laplacien scalaire	$\Delta f = \mathrm{div}(\overrightarrow{grad}\, f)$
laplacien vectoriel	$\Delta\vec{A} = \overrightarrow{grad}(\mathrm{div}\, \vec{A}) - \overrightarrow{rot}(\overrightarrow{rot}\, \vec{A})$

I.1. Coordonnées cartésiennes

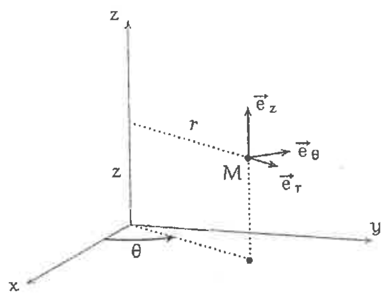
◇ $\overrightarrow{grad}\, f = \dfrac{\partial f}{\partial x}\vec{e}_x + \dfrac{\partial f}{\partial y}\vec{e}_y + \dfrac{\partial f}{\partial z}\vec{e}_z$

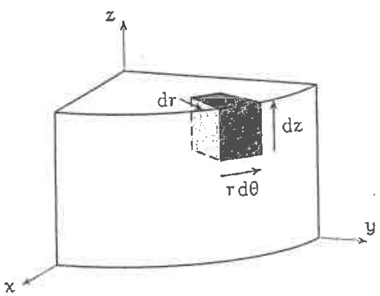
◇ $\mathrm{div}\, \vec{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

◇ $\overrightarrow{rot}\, \vec{A} = \begin{bmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{bmatrix}$

◇ $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

I.2. Coordonnées cylindriques





$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\begin{cases} r \geqslant 0 \\ 0 \leqslant \theta < 2\pi \\ -\infty \leqslant z \leqslant +\infty \end{cases}$$

◇ $\overrightarrow{grad}\, f = \dfrac{\partial f}{\partial r}\vec{e}_r + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\vec{e}_\theta + \dfrac{\partial f}{\partial z}\vec{e}_z$

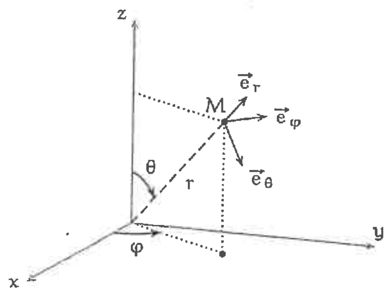
◇ $\mathrm{div}\, \vec{A} = \dfrac{1}{r}\dfrac{\partial (r A_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial A_\theta}{\partial \theta} + \dfrac{\partial A_z}{\partial z}$

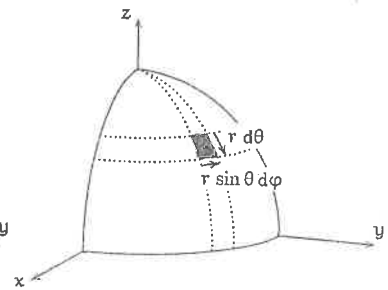
◇ $\overrightarrow{rot}\, \vec{A} = \left[\dfrac{1}{r}\dfrac{\partial A_z}{\partial \theta} - \dfrac{\partial A_\theta}{\partial z}\right]\vec{e}_r + \left[\dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r}\right]\vec{e}_\theta + \left[\dfrac{1}{r}\dfrac{\partial (r A_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial A_r}{\partial \theta}\right]\vec{e}_z$

◇ $\Delta f = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \times \dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \theta^2} + \dfrac{\partial^2 f}{\partial z^2}$

◇ $\Delta\vec{A} = \begin{bmatrix} \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \theta^2} + \frac{\partial^2 A_r}{\partial z^2} + \frac{1}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} - \frac{A_r}{r^2} \\ \frac{\partial^2 A_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial A_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2} \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial z^2} \end{bmatrix}$

I.3. Coordonnées sphériques





$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} r \geqslant 0 \\ 0 \leqslant \varphi < 2\pi \\ 0 \leqslant \theta \leqslant \pi \end{cases}$$

◇ $\overrightarrow{grad}\, f = \dfrac{\partial f}{\partial r}\vec{e}_r + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\vec{e}_\theta + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \varphi}\vec{e}_\varphi$

◇ $\mathrm{div}\, \vec{A} = \dfrac{1}{r^2}\dfrac{\partial (r^2 A_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial (\sin\theta\, A_\theta)}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial \varphi}$

◇ $\overrightarrow{rot}\, \vec{A} = \dfrac{1}{r\sin\theta}\left[\dfrac{\partial (\sin\theta\, A_\varphi)}{\partial \theta} - \dfrac{\partial A_\theta}{\partial \varphi}\right]\vec{e}_r +$

$\dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial A_r}{\partial \varphi} - \dfrac{\partial (r A_\varphi)}{\partial r}\right]\vec{e}_\theta + \dfrac{1}{r}\left[\dfrac{\partial (r A_\theta)}{\partial r} - \dfrac{\partial A_r}{\partial \theta}\right]\vec{e}_\varphi$

◇ $\Delta f = \dfrac{1}{r}\dfrac{\partial^2 (rf)}{\partial r^2} + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial f}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial \varphi^2}$

II. Identités vectorielles

◇ $\mathrm{div}(\overrightarrow{grad}\, f) = \Delta f$

◇ $\mathrm{div}(\overrightarrow{rot}\, \vec{A}) = 0$

◇ $\overrightarrow{rot}(\overrightarrow{grad}\, f) = \vec{0}$

◇ $\overrightarrow{rot}(\overrightarrow{rot}\, \vec{A}) = \overrightarrow{grad}\,(\mathrm{div}\, \vec{A}) - \Delta\vec{A}$

◇ $\overrightarrow{grad}(f \cdot g) = f \cdot \overrightarrow{grad}\, g + g \cdot \overrightarrow{grad}\, f$

◇ $\mathrm{div}(f \cdot \vec{A}) = f \cdot \mathrm{div}\, \vec{A} + \vec{A} \cdot \overrightarrow{grad}\, f$

◇ $\mathrm{div}(\vec{A} \wedge \vec{B}) = \vec{B} \cdot \overrightarrow{rot}\, \vec{A} - \vec{A} \cdot \overrightarrow{rot}\, \vec{B}$

◇ $\overrightarrow{rot}(f \cdot \vec{A}) = f \cdot \overrightarrow{rot}\, \vec{A} + (\overrightarrow{grad}\, f) \wedge \vec{A}$